Some properties of nonlinear model predictive control with guaranteed nominal stability

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***Abstract*—In this note, we discuss the properties of model predictive control of nonlinear systems with input constraints. It shows that the prediction trajectory will not leave the terminal set once it enters into it, and the terminal state lies in a sublevel set of the terminal set if there exists a point of the prediction trajectory lying in the sublevel set. Furthermore, we show that the feasible set of the related optimization problem is a bounded set around the origin.**

I. Introduction

Model predictive control (MPC) is an effective strategy to deal with multivariable control problems of constrained nonlinear systems. At each time instant, a control sequence is obtained by solving an optimization problem, where the current state of the plant is adopted as the initial state. Only the first control action in this sequence is applied to the plant. Both stability and robustness of MPC are well developed [1], [2], [3] in the last three decades. Furthermore, applications of MPC have spanned from process control [4] to nonholonomic mobile robots [5], aerospace [6], and transportation networks [7].

Generally, MPC with guaranteed nominal stability needs to calculate a terminal cost, a terminal set, and a terminal control law off-line [1], [8]. The system state will be driven to the terminal set in finite time, and the terminal cost is an upper bound of the cost function for the system state in the terminal set. MPC of nonlinear systems with input constraints has some degree of inherent robustness with respect to persistent but bounded disturbances when the terminal control law and the terminal penalty matrix are chosen as the linear quadratic control law and the related Lyapunov matrix, respectively [9].

In this note, we will further discuss some properties of model predictive control of nonlinear systems with input constraints. It shows that the prediction trajectory will not leave the terminal set once it enters into it, and the terminal state will lie in a sublevel set of the terminal set if there exists a point of the prediction trajectory which lies in the set. Furthermore, we show that the feasible set of the related optimization problem is a bounded set.

The remainder of the note is organized as follows. In Section II, MPC with guaranteed nominal stability is brief introduced. The properties of prediction trajectory, and the boundedness of the feasible set are discussed in Section III and Section IV, respectively. Section V concludes the paper with a short summary.

Notation: For simplicity, we denote $\|x\|_P^2 := x^T P x$, where P is a symmetric positive definite matrix.

II. Preliminaries

Consider nonlinear continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the system state and $u(t) \in \mathbb{R}^{n_u}$ the control input at time instant t.

The system is subject to the control constraint

$$u(t) \in \mathcal{U}, \tag{2}$$

where the set $\mathcal{U} \subset \mathbb{R}^{n_u}$ is a compact set which contains $0 \in \mathbb{R}^{n_u}$ in its interior.

Some standing assumptions are stated as follows:

Assumption 1: The system state x can be measured instantaneously.

Assumption 2: f is twice continuously differentiable, and $f(0,0) = 0$. Thus, $0 \in \mathbb{R}^{n_x}$ is an equilibrium of the nominal system.

Assumption 3: The system (1) has a unique solution for any initial condition x_0 and any piecewise right-continuous input function $u(\cdot) : [0, T_p] \rightarrow \mathcal{U}$, where $T_p > 0$ is a given constant. The optimization problem is formulated as follows:

Problem 1:

$$\begin{aligned} & \underset{u(\cdot)}{\text{minimize}} \ J(x(t), u(\cdot)) \\ \text{subject to} \quad & \dot{x}(\tau, x(t)) = f(x(\tau, x(t)), u(\tau)), \\ & x(t, x(t)) = x(t), \\ & u(\tau) \in \mathcal{U}, \ \tau \in [t, t+T_p], \\ & x(t+T_p, x(t)) \in \mathcal{X}_f, \end{aligned}$$

where

$$J(x(t), u(\cdot)) := \|x(t+T_p, x(t))\|_P^2 + \int_t^{t+T_p} \left(\|x(s, x(t))\|_Q^2 + \|u(s)\|_R^2 \right) ds,$$

is the cost functional, T_p is the prediction horizon, $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive definite state and input weighting matrices. The positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ is the terminal penalty matrix, and $E(x) := \|x\|_P^2$ is the terminal penalty function. The terminal set $\mathcal{X}_f := \left\{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \alpha\right\}$ is a sublevel set of the terminal penalty function. The term $x(\cdot, x(t))$ represents the predicted state trajectory starting from the initial state $x(t)$ under the control $u(\cdot)$.

The set $\mathcal{X}_f$ and the function $E(x)$ are said to be the terminal region and the terminal penalty respectively, if there exists a regional control law $u = Kx$ such that the following conditions are satisfied [1], [8]:

B0) $Kx \in \mathcal{U}$, for all $x \in \mathcal{X}_f$,

B1) $E(x)$ satisfies inequality

$$\frac{\partial E(x)}{\partial x} f(x, Kx) + x^T (Q + K^T R K) x \leq 0, \forall x \in \mathcal{X}_f. \tag{3}$$

The terminal set $\mathcal{X}_f$ has the following properties [10]:

- The point $0 \in \mathcal{R}^{n_x}$ is contained in the interior of $\mathcal{X}_f$ due to the positive definiteness of $E(x)$ and $\alpha > 0$,
- In terms of (3), the terminal set $\mathcal{X}_f$ is invariant for the nonlinear system (1) with the local control $u = Kx$.

According to the principle of MPC, the optimization problem will be solved at the sampling instants $t_j = j\delta$, where δ is a sampling time and $0 < \delta \leq T_p$, $j \in \mathbb{Z}_{[0,\infty)}$.

Assuming that the minimum is attained, the optimal solution to Problem 1 is given by the optimal input trajectory,

$$u^*(\tau, x(t)) := \arg \min_{\substack{u(\cdot) \in \mathcal{U} \\ x(t+T_p, x(t)) \in \mathcal{X}_f}} J(x(t), u(\tau)),$$

for all $\tau \in [t, t + T_p]$. The applied control is $u^*(\tau, x(t))$, for all $\tau \in [t, t + \delta)$.

The following stability results was established [8], [10]:

Lemma 1: Suppose that

(a) Assumptions 1-3 are satisfied,

(b) there exist an asymptotically stable control law $u = Kx$, a continuously differentiable, positive definite function $E(x)$ that satisfies (3) for all $x \in \mathcal{X}_f$,

(c) Problem 1 is feasible at the initial time instant $t = 0$.

Then,

i) the open-loop optimal control problem is feasible for all time $t \geq 0$,

ii) the system under the MPC control law is nominally asymptotically stable

III. Properties of the Prediction Trajectory

The next two lemmas show that along the prediction trajectory, the terminal state is the closest point to the origin in the sense of $\|x\|_p$, and the prediction trajectory will never leave the terminal set once it enters into it.

Lemma 2: Let $\varpi \in (0, 1]$. Define a subset of the terminal set $\mathcal{X}_f$,

$$\mathcal{X}_f^{\varpi} := \left\{x \in \mathbb{R}^{n_x} \mid x^T P x \leq \varpi\alpha\right\}.$$

Then, the terminal state is in the sublevel set $\mathcal{X}_f^{\varpi}$ if there exists a point of the predicted trajectory which lies in the sublevel set $\mathcal{X}_f^{\varpi}$.

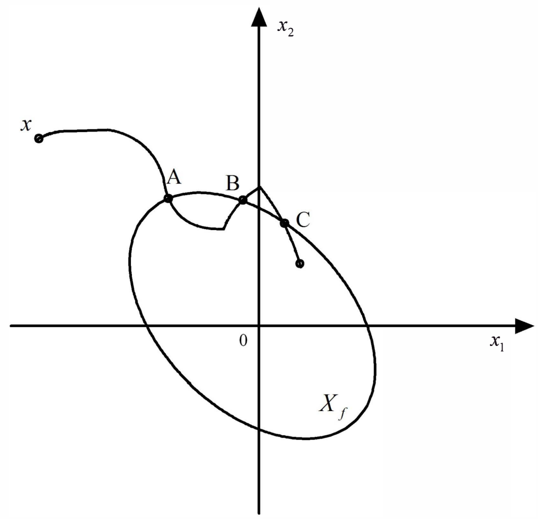

Fig. 1. Fictitious prediction trajectory: enter into the terminal set, leave it for a while and come back in the end

Proof: Suppose that the initial state is $x(t_0)$ and there exists $x(t, x(t_0)) \in \mathcal{X}_f^{\varpi} \subseteq \mathcal{X}_f$ with $t \in [t_0, t_0 + T_p]$, i.e. $x(t, x(t_0))^T P x(t, x(t_0)) \leq \varpi\alpha$. The linear control law Kx guarantees that

$$\frac{dE(x)}{dt} \leq -x^T (Q + K^T R K) x, \quad \forall x \in \mathcal{X}_f.$$

Integrating the above inequality from t_0 to $t_0 + T_p$, yields

$$E(x(t_0 + T_p, x(t_0))) - E(x(t, x(t_0))) \leq -\int_{t_0}^{t_0+T_p} \|x(\tau, x(t_0))\|_{Q+K^T RK}^2 d\tau.$$

That is,

$$E(x(t, x(t_0))) \geq E(x(t_0 + T_p, x(t_0))) + \int_{t_0}^{t_0+T_p} \|x(\tau, x(t_0))\|_{Q+K^T RK}^2 d\tau.$$

Since the linear control law Kx is a feasible solution to Problem 1, we have by the Principle of Optimality,

$$E(x(t, x(t_0))) \geq \int_{t_0}^{t_0+T_p} \left(\|x^*(\tau, x(t_0))\|_Q^2 + \|u^*(\tau, x(t_0))\|_R^2 d\right)\tau + E(x^*(t_0 + T_p, x(t_0))), \tag{4}$$

where $x^*(\tau, x(t_0))$ and $u^*(\tau, x(t_0))$ denote the optimal predicted state and the optimal predicted control, respectively.

Due to (4), and $x(t, x(t_0)) \in \mathcal{X}_f^{\varpi} \subseteq \mathcal{X}_f$, we have

$$\begin{aligned} \varpi\alpha \geq & E(x(t, x(t_0))) \\ > & E(x^*(t_0 + T_p, x(t_0))). \end{aligned}$$

Thus, $x^*(t_0 + T_p, x(t_0)) \in \mathcal{X}_f^{\varpi}$. □

Lemma 3: the prediction trajectory will never leave the terminal set $\mathcal{X}_f$ once it enters into it.

Proof: For the sake of contradiction, assume the prediction trajectory will leave the terminal set after it enters into it, see Fig. 1.

For simplicity, denote the time instants that the prediction trajectory go across the points A, B and C as $t+t_A$, $t+t_B$ and $t+t_C$, respectively.

Since the terminal control law is a feasible solution to the optimization problem for the states in the terminal set, similar to (4), we have

$$E(x(t+t_C, x(t))) \geq E(x^*(t+T_p, x(t)))+ \int_{t+t_C}^{t+T_p} \left(\|x^*(\tau, x(t))\|_Q^2 + \|u^*(\tau, x(t))\|_R^2 d\tau\right),$$

and

$$E(x(t+t_B, x(t))) \geq E(x^*(t+t_C, x(t)))+ \int_{t+t_B}^{t+t_C} \left(\|x^*(\tau, x(t))\|_Q^2 + \|u^*(\tau, x(t))\|_R^2 d\tau\right).$$

Thus,

$$\begin{aligned} J(x(t)) &= \int_t^{t+T_p} \left(\|x^*(\tau, x(t))\|_Q^2 + \|u^*(\tau, x(t))\|_R^2\right) d\tau \\ &\quad + E(x^*(t+T_p, x(t))) \\ &\leq \int_t^{t+t_C} \left(\|x^*(\tau, x(t))\|_Q^2 + \|u^*(\tau, x(t))\|_R^2\right) d\tau \\ &\quad + E(x^*(t+t_C, x(t))) \\ &\leq \int_t^{t+t_B} \left(\|x^*(\tau, x(t))\|_Q + \|u^*(\tau, x(t))\|_R^2\right) d\tau \\ &\quad + E(x^*(t+t_B, x(t))). \end{aligned}$$

Since the points B and C are in the boundary of the set $\mathcal{X}_f$,

$$\begin{aligned} E(x^*(t+t_B, x(t))) &= E(x^*(t+t_C, x(t))) \\ &= \alpha. \end{aligned}$$

Thus,

$$\int_{t+t_B}^{t+t_C} \left(\|x^*(\tau, x(t))\|_Q^2 + \|u^*(\tau, x(t))\|_R^2\right) d\tau \leq 0,$$

which contradicts with the fact that $x^T Q x > 0$ for all $x \neq 0$, and $t_B \neq t_C$. □

IV. Boundedness of the Feasible Set

Feasibility of the optimization problem means that there exists at least one input function $u(\tau) \in \mathcal{U}$, with $\tau \in [t, t+T_p]$, such that the value of the objective function is finite and the terminal constraint is satisfied.

Let r be a given constant. The set $\mathcal{B}(r)$ is convex and compact. Since $f(\cdot,\cdot)$ is twice continuous differentiable on $\mathcal{B}(r) \times \mathcal{U}$, and $\mathcal{U}$ is a compact set, $[\partial f/\partial x]$ is bounded on $\mathcal{B}(r) \times \mathcal{U}$. That is, $\|\partial f/\partial x\|$ is bounded on $\mathcal{B}(r) \times \mathcal{U}$.

Lemma 4: Let $v \geq 0$ be a constant such that

$$\left\|\frac{\partial f}{\partial x}(x, u)\right\| \leq v.$$

on $\mathcal{B}(r) \times \mathcal{U}$. Then,

$$\|f(y, u) - f(x, u)\| \leq v\|y - x\|. \tag{5}$$

Proof: Fixed $u \in \mathcal{U}$, $x \in \mathcal{B}(r)$ and $y \in \mathcal{B}(r)$. Defined $\varrho(s) := (1-s)x + sy$ for $0 \leq s \leq 1$. Since $\mathcal{B}(r)$ is convex, $\varrho(s) \in \mathcal{B}(r)$. Take $z \in \mathbb{R}^{x_x}$ such that $\|z\| = 1$ and

$$z^T[f(y, u) - f(x, u)] = \|f(y, u) - f(x, u)\|.$$

Set $g(s) := z^T f(\varrho(s), u)$. Since $\varrho(s)$ is a real-valued function, which is continuous differentiable in an open interval that includes $[0, 1]$, we conclude by the mean value theorem that there is $s_1 \in [0, 1]$ such that

$$g(1) - g(0) = \dot{g}(s_1)$$

Evaluating g at $s = 0$ and $s = 1$, and calculating $\dot{g}(s)$ by using the chain rule, we obtain

$$\begin{aligned} z^T[f(y, u) - f(x, u)] &= z^T \frac{\partial f}{\partial x}(\varrho(s_1), u)(y - x) \\ \|f(y, u) - f(x, u)\| &\leq \|z\| \cdot \|\frac{\partial f}{\partial x}(\varrho(s_1), u)\| \cdot \|y - x\| \\ &\leq v\|y - x\|. \end{aligned}$$

□

In order to show the property of $\mathcal{X}_f$, we will introduce a lemma which is a variant of Gronwall-Bellman Inequality [11].

Lemma 5: Let $\lambda_h : [a, b] \to \mathbb{R}$ and $\mu_h : [a, b] \to \mathbb{R}$ be continuous and nonnegative. If a continuous function $y : [a, b] \to \mathbb{R}$ satisfies

$$y(t) \leq \lambda_h(t) + \int_t^b \mu_h(s) y(s) ds$$

for $a \leq t \leq b$, then in the same interval

$$y(t) \leq \lambda_h(t) + \int_t^b \lambda_h(s)\mu_h(s) \exp\left[-\int_s^t \mu_h(\tau) d\tau\right] ds$$

Proof: let $z(t) = \int_t^b \mu_h(s) y(s) ds$, and $v(t) = z(t) + \lambda_h(t) - y(t) \geq 0$. Then, z is differentiable and

$$\dot{z} = -\mu_h(t) y(t) = -\mu_h(t)[z(t) + \lambda_h(t) - v(t)]$$

This is a scalar linear state equation with the transition function

$$\Phi(t, s) = \exp\left[-\int_s^t \mu_h(\tau) d\tau\right].$$

Since $z(b) = 0$, we have

$$z(t) = \int_t^b \Phi(t, s)[\mu_h(s)\lambda_h(s) - \mu_h(s) v(s)] ds.$$

The term

$$\int_t^b \Phi(t, s)\mu_h(s) v(s) ds$$

is nonnegative. Therefore,

$$\begin{aligned} z(t) &\leq \int_t^b \Phi(t, s)\mu_h(s)\lambda_h(s) ds \\ &= \int_t^b e^{-\int_s^t \mu_h(\tau) d\tau} \mu_h(s)\lambda_h(s) ds \end{aligned}$$

Since $y(t) \leq \lambda_h(t) + z(t)$,

$$y(t) \leq \lambda_h(t) + \int_t^b e^{-\int_s^t \mu_h(\tau)d\tau} \mu_h(s)\lambda_h(s)ds.$$

□

We can now show the following crucial property of $\mathcal{X}_r$.

Lemma 6: The set $\mathcal{X}_r$ is bounded.

Proof: Let $x_0 \in \mathcal{X}_r$ be an initial state of the nominal system (1). For simplicity, denote the corresponding control trajectory as $u(\tau)$, $\tau \in [0, T_p]$.

Denote that $x(t, x_0)$, $t \in [0, T_p]$, is a prediction trajectory of

$$\dot{x} = f(x, u), \quad x(t_0) = x_0$$

with the fixed $u \in \mathcal{U}$. Thus,

$$x(t, x_0) = x_0 + \int_0^t f(x(s), u(s))ds$$

Denote $x(T_p) := x(T_p, x_0)$. The trajectory $x(t, x_0)$, $t \in [0, T_p]$, can be rewritten as

$$\begin{aligned} x(t, x_0) =& x_0 + \int_0^{T_p} f(x(s, x_0), u(s))ds \\ &- \int_t^{T_p} f(x(s, x_0), u(s))ds \\ =& x(T_p) - \int_t^{T_p} f(x(s), u(s))ds \end{aligned}$$

Then,

$$\begin{aligned} & x(t, x_0) - x(T_p) = - \int_t^{T_p} f(x(s, x_0), u(s))ds \\ =& \int_t^{T_p} [-f(x(s, x_0), u(s)) + f(x(T_p), u(s)) \\ & - f(x(T_p), u(s))]ds \end{aligned}$$

That is,

$$\begin{aligned} & \|x(t, x_0) - x(T_p)\| \\ \leq & \int_t^{T_p} \|f(x(s, x_0), u(s)) - f(x(T_p), u(s))\|ds \\ & + \int_t^{T_p} \|f(x(T_p), u(s))\|ds. \end{aligned}$$

Let

$$h_0 := \|f(x(T_p), u(s))\|.$$

Since for fixed $u(s) \in \mathcal{U}$, $\|f(x(s, x_0), u(s)) - f(x(T_p), u(s))\| \leq v\|x(s, x_0) - x(T_p)\|$, we obtain

$$\|x(t, x_0) - x(T_p)\| \leq \int_t^{T_p} v\|x(s, x_0) - x(T_p)\|ds + h_0(T_p - t).$$

Application of the Lemma 5 to the function $x(t, x_0) - x(T_p)$ results in

$$\|x(t, x_0) - x(T_p)\| \leq \int_t^{T_p} vh_0(T_p - s)e^{-v(t-s)}ds + h_0(T_p - t).$$

Integrating the right-hand side, we obtain

$$\|x(t, x_0) - x(T_p)\| \leq \frac{h_0}{v}\left(e^{vT_p} - e^{vt}\right).$$

Since $x_0 = x(0, x_0)$ and $x(t_0 + T_p, x(t_0)) \in \mathcal{X}_f$, it directly follows

$$\begin{aligned} \|x_0\| \leq & \|x(T_p)\| + \|x(0, x_0) - x(T_p)\| \\ \leq & \frac{\alpha}{\lambda_{\min}(P)} + \frac{h_0}{v}(e^{vT_p} - 1). \end{aligned}$$

Denote

$$h := \max_{\substack{u \in \mathcal{U} \\ x \in \mathcal{X}_f}} \|f(x, u)\|.$$

Since $x(T_p) \in \mathcal{X}_f$ and $u(s) \in \mathcal{U}$, and $h_0 \leq h$, we have

$$\|x_0\| \leq \frac{\alpha}{\lambda_{\min}(P)} + \frac{h}{v}(e^{vT_p} - 1).$$

This complete the proof. □

V. Conclusion

In this note, we showed that the properties of model predictive control of nonlinear systems with input constraints, which include (1) the terminal state lies in a sublevel set of the terminal set if there exists a point of the prediction trajectory lying in the sublevel set, (2) the prediction trajectory will not leave the terminal set once it enters into the terminal set, and (3) the feasible set of the corresponding optimization problem is a bounded set.

Acknowledgment

Shuyou Yu, Ting Qu and Hong Chen gratefully acknowledge support by the 973 Program (No. 2012CB821202), the National Nature Science Foundation of China (No. 61034001), and the Program for Changjiang Scholars and Innovative Research Team in University (No. IRT1017).

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